Fuzzy Topological B-algebras

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Abstract

In this note the notion of fuzzy topological B-algebras is introduced. The Foster's results on homomorphic images and inverse images in fuzzy topological B-algebras are studied.

Keywords: (fuzzy) B-algebra, fuzzy topological B-algebras.

1. Introduction

Y. Imai and K. Iseki [4] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [7] J. Neggers and H. S. Kim introduced the notion of d-algebras, which is generalization of BCK-algebras and investigated relation between d-algebras and BCK-algebras. Also they introduced the notion of B-algebras [6], which is a generalization of BCK-algebra. Y. B. Jun et. al. applied the fuzzy notions to B-algebras and introduced the notions of fuzzy B-algebras [3], and present author introduce the notion Interval-valued fuzzy B-algebras [1], which is generalization of fuzzy B-algebras.

The concept of a fuzzy set, which was introduced in [9]. Provides a natural framework for generalizing many of the concepts of general topology to what might be called fuzzy topological spaces. D. H. Foster (cf. [2]) combined the structure of a fuzzy topological space with that of a fuzzy group, introduced by A. Rosenfeld (cf. [8]), and to formulated the elements of a theory of fuzzy topological groups. In the present paper, we introduced the concept of fuzzy topological B-algebras and apply some of Fosters results on homomorphic images and inverse images to fuzzy topological B-algebras.

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2. Preliminary Notes

Definition 2.1. [6] A B-algebra is a non-empty set X with a consonant 0 and a binary operation $*$ satisfying the following axioms:

(I) $x * x = 0$,
(II) $x * 0 = 0$,
(III) $(x * y) * z = x * (z * (0 * y))$,
For all $x, y, z \in X$.

Example 2.2. [3] Let $X = \{0,1,2,3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

Then we easily can check that $(X,*,0)$ is a B-algebra, since we have $x * x = 0$, $x * 0 = 0$ and $(x * y) * z = x * (z * (0 * y))$, for all $x, y, z \in X$. But $(X,*,0)$ is not a BCK-algebra, since $0 * 1 \neq 0$.

Theorem 2.3. [6] In a B-algebra X, we have $x * y = x * (0 * (0 * y))$, for all $x, y \in X$,

Definition 2.4. A non-empty subset I of a B-algebra X is called sub algebra of X if $x * y \in I$ for any $x, y \in X$.

A mapping $f : X \to Y$ of B-algebras is called a B-homomorphism if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$.

We now review some fuzzy logic concept (see [9]). Let X be a set. A fuzzy set A in X is characterized by a membership function $\mu_A : X \to [0,1]$. Let f be a mapping from the set X to the set Y and let B be a fuzzy set in Y with membership function μ_B.

The inverse image of B, denoted $f^{-1}(B)$, and is the fuzzy set in X with membership function $\mu_{f^{-1}(B)}$ defined by $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$ for all $x \in X$.

Conversely, let A be a fuzzy set in X with membership function μ_A Then the image of A, denoted by $f(A)$, is the fuzzy set in Y such that:

$$\mu_{f(A)}(y) = \begin{cases} \sup\limits_{z \in f^{-1}(y)} \mu_A(z) & f^{-1}(y) \neq \phi \\ 0 & otherwise \end{cases}$$

Where $f^{-1} = (y) = \{x | f(x) = y\}$.

Definition 2.5. A fuzzy set A in the B-algebra X with the membership function μ_A is said to be have the sup property if for any subset $T \subset X$ there exists $x_0 \in T$ such that

$$\mu_A(x_0) = \sup_{t \in T} \mu_A(t)$$

Definition 2.6. A fuzzy topology on a set X is a family τ of fuzzy sets in X which satisfies the following condition :

(i) For $c \in [0,1]$, $K_c \in \tau$, where K_c has a constant membership function,

(ii) If $A, B \in \tau$, then $A \cap B \in \tau$,

(iii) τ closed under arbitrary union, which means that if $A_j \in \tau$ for all $j \in J$ J, then $\bigcup_{j \in J} A_j \in \tau$.

The pair (X, τ) is called a fuzzy topological space and members of τ are called open fuzzy sets.

Definition 2.7. Let A be a fuzzy set in X and τ a fuzzy topology on X . Then the induced fuzzy topology on A is the family of fuzzy subsets of A which are the intersection with A of τ-open fuzzy sets in X . The induced fuzzy topology is denoted by τ_A, and the pair (X, τ_A) is called a fuzzy subspace of (X, τ).

Definition 2.8. Let (X, τ) and (Y, υ) be two fuzzy topological spaces. A mapping f of (X, τ) into (Y, υ) is fuzzy continuous if for each open fuzzy set V in υ the inverse image $f^{-1}(V)$ is in τ .

Conversely, f is fuzzy open if for each fuzzy set V in τ, the image $f(V)$ is in υ.

Let (A, τ_A) and (B, υ_B) be fuzzy subspace of fuzzy topological spaces (X, τ) and (Y, υ) respectively, and let f be a mapping from (X, τ) to (Y, υ). Then f is a mapping of (A, τ_A) into (B, υ_B) if $f(A) \subseteq B$. Furthermore f is relatively fuzzy continuous if for each open fuzzy set V' in υ_B the intersection $f^{-1}(V') \cap A$ is in τ_A. Conversely, f is relatively fuzzy open if for each open fuzzy set U', the image $f(U')$ is in υ_B.

Lemma 2.9. [2] Let (A, τ_A), (B, υ_B) be fuzzy subspace of fuzzy topological space (X, τ), (Y, υ) respectively, and let f be a fuzzy continuous mapping of (X, τ) into (Y, υ) such that $f(A) \subset B$ Then f is a relatively fuzzy continuous mapping of (A, τ_A) into (B, υ_B).

3. Fuzzy topological B-algebra

From now on X is a B-algebra, unless otherwise is stated.

Definition 3.1. [3] Let μ be a fuzzy set in a B-algebra. Then μ is called a fuzzy B-algebra (sub algebra) of X if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in X$.

Example 3.2. (a) Let $X = \{0,1,2,3,4,5\}$ be a set with the following table:

$*$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Then X is a B-algebra. Define a fuzzy set $\mu: X \to [0,1]$ by μ (0) = μ (3) = 0.7 > 0.1 = μ(x) for all $x \in X \setminus \{0,3\}$. Then μ is a fuzzy B-sub-algebra of X [3].

(b) Let Z be the group of integers under usual addition and let $\alpha \notin Z$. We adjoin the special element α to Z. Let $X := Z \cup \{\alpha\}$. Define $\alpha + 0 = \alpha$, $\alpha + n = n - 1$, where $n \neq 0$ in Z and $\alpha + \alpha$ is an arbitrary element in X. Define a mapping $\varphi: X \to X$ by $\varphi(\alpha) = 1$, $\varphi(n) = -n$ where $n \in Z$. If we define a binary operation "*" on X by $x * y = x + \varphi(y)$, then $(X, *, 0)$ is a B-algebra.
Now define $\mu: X \to [0,1]$ as follows:

$$\mu(x) = \begin{cases} \left|\frac{1}{x}\right|, x \neq 0 \\ 1, x = \alpha, 0 \end{cases}$$

Then it is clear that μ is a fuzzy B-algebra that has sup property [1].

Proposition 3.3. Let f be a B-homomorphism from X into Y and G is a fuzzy B-algebra of Y with the membership function μ_G. Then the inverse image $f^{-1}(G)$ of G is a fuzzy B-algebra of X.

Proof. Let $x, y \in X$. The

$$\begin{aligned} \mu_{f^{-1}(G)}(x * y) &= \mu_G(f(x * y)) \\ &= \mu_G(f(x) * f(y)) \\ &\geq \min\{(\mu_G(f(x), \mu_G(f(y))\} \\ &= \min\{\mu_{f^{-1}(G)}(x), \mu_{f^{-1}(G)}(y)\}. \end{aligned}$$

Proposition 3.4. Let f be a B-homomorphism from X onto Y and D is a fuzzy B-algebra of X with the sup property. Then the image f (D) of D is a fuzzy B-algebra of Y.

Proof. Let $a, b \in Y$, let $x_0 \in f^{-1}(a)$, $y_0 \in f^{-1}(b)$ such that:

$$\mu_D(y_0) = \sup_{t \in f^{-1}(b)} \mu_D(t) \text{ and}$$

$$\mu_D(x_0) = \sup_{t \in f^{-1}(a)} \mu_D(t)$$

Then by the definition of $\mu_{f(D)}$, we have

$$\begin{aligned} \mu_{f(D)}(x * y) &= \sup_{t \in f^{-1}(a*b)} \mu_D(t) \\ &\geq \mu_D(x_0 * y_0) \\ &\geq \min\{\mu_D(x_0), \mu_D(y_0)\} \\ &= \min\{\sup_{t \in f^{-1}(b)} \mu_D(t), \sup_{t \in f^{-1}(a)} \mu_D(t)\} \\ &= \min\{\mu_{f(D)}(a), \mu_{f(D)}(b)\}. \end{aligned}$$

For any B-algebra X and any element $a \in X$ we denote by R_a the right translation of X defined by $R_a(x) = x * a$ for all $x \in X$. It is clear that $R_0(x) = 0 = R_x(0)$ For all $x \in X$.

Definition 3.5. Let τ be a fuzzy topology on X and D be a fuzzy B-algebra of X with induced topology τ_D. Then D is called a fuzzy topological B-algebra of X if for each $a \in X$ the mapping $R_a: (D, \tau_D) \to (D, \tau_D)$ is relatively fuzzy continuous.

Example 3.6. In Example 3.2 (a), consider fuzzy set A in X defined by:

$$A(x) = \begin{cases} 1 & x = 0, \\ 0.7 & x = 1, \\ 0.6 & x = 2, \\ 08 & x = 3, \\ 0.3 & x = 4, \\ 0.1 & x = 5 \end{cases}$$

Then $\tau = \{\bar{0}, A, \bar{1}\}$ is a fuzzy topology on X, where $\bar{0}(x) = 0$ and $\bar{1}(x) = 1$ for all $x \in X$. Now, consider fuzzy B-sub-algebra D = μ, defined in Example 3.2 (a). Then $\tau_D = \{\bar{0}, A \cap D, \bar{1}\}$ is relative fuzzy topology on X and the mapping $R_a: (D, \tau_D) \to (D, \tau_D)$ is relatively fuzzy continuous.

Theorem 3.7. Let X and Y be two B-algebras, $f: X \to Y$ be a B-homomorphism. Let τ and ν be the fuzzy topologies on X and Y respectively, such that $\tau = f^{-1}(\nu)$. Let G be a fuzzy topological B-algebra of Y with membership function μ_G. Then $f^{-1}(G)$ is a fuzzy topological B-algebra of X with membership function $\mu_{f^{-1}(G)}$.

Proof. We must show that, for each $a \in X$, the mapping

$R_a : (f^{-1}(G), \tau_{f^{-1}(G)}) \to (f^{-1}(G), \tau_{f^{-1}(G)})$

is relatively fuzzy continuous. Let U be any open fuzzy set in $\tau_{f^{-1}(G)}$ on $f^{-1}(G)$. Since f is a fuzzy continuous mapping from (X,τ) into (Y,υ), from Lemma 2.9 follows that f is a relatively fuzzy continuous mapping of $(f^{-1}(G), \tau_{f^{-1}(G)})$ into (G, ν_G). Note that there exists an open fuzzy set V in ν_G such that $f(V) = U$. The membership function of $R_a^{-1}(U)$ is given by

$\mu_{R_a^{-1}(U)}(x) = \mu_U(R_a(x))$
$= \mu_U(x * a)$
$= \mu_{f^{-1}(V)}(x * a)$
$= \mu_V(f(x * a))$
$= \mu_V(f(x) * f(a))$.

Since G is a fuzzy topological B-algebra of Y, the mapping $R_b : (G, \tau_G) \to (G, \tau_G)$ is relatively fuzzy continuous for each $b \in Y$. Hence

$\mu_{R_a^{-1}(U)}(x) = \mu_V(f(x) * f(a))$
$= \mu_V(R_{f(a)} f(x))$
$= \mu_V(R_{f(a)} f(x))$
$= \mu_{R_{f(a)}^{-1}(V)}(f(x))$
$= \mu_{f^{-1}(R_{f(a)}^{-1}(V))} R_{f(a)}(x)$.

which implies that $R_a^{-1}(U) = f^{-1}(R_{f(a)}^{-1}(V))$. Therefore $R_a^{-1}(U) \cap f^{-1}(G) = f^{-1}(R_{f(a)}^{-1}(V)) \cap f^{-1}(G)$ is an open in the relative fuzzy topology on $f^{-1}(G)$.

The membership function μ_G of a fuzzy B-algebra G of X is said to be f-invariant [8] if f(x) = f(y) implies $\mu_G(x) = \mu_G(y)$, for all $x, y \in X$.

Theorem 3.8. Given B-algebras X and Y and a B-homomorphism f from X onto Y, let τ be the fuzzy topology on X and ν be the fuzzy topology on Y such that $f(\tau) = \nu$. Let D be a fuzzy topological B-algebra of X. If the membership function μ_D of D is a f-invariant, then f(D) is a fuzzy topological B-algebra of Y.

Proof. It is enough to show that the mapping $R_b : (f(D), \nu_{f(D)}) \to (f(D), \nu_{f(D)})$ is relatively fuzzy continuous, for all $b \in Y$. It is clear that f is a relatively fuzzy open mapping, since for $U \in \tau_D$ there exists $U' \in \tau$ such that $U = U' \cap D$, by f-invariance of μ_D we have $f(U) = f(U) \cap f(D) \in \nu_{f(D)}$.

Let V' be an open fuzzy set in $\nu_{f(D)}$. For any $b \in Y$ by hypothesis there exists $a \in X$ such that $b = f(a)$. Thus

$\mu_{f^{-1}(R_b^{-1}(V'))}(x) = \mu_{f^{-1}(R_{f(a)}^{-1}(V'))}(x)$
$= \mu_{R_{f(a)}^{-1}(V')}(f(x))$
$= \mu_{V'}(R_{f(a)}(f(x))$
$= \mu_{V'}(f(x) * f(a))$
$= \mu_{V'}(f(x * a))$
$= \mu_{f^{-1}(V')}(x * a)$
$= \mu_{f^{-1}(V')}(R_a(x))$
$= \mu_{R_a^{-1} f^{-1}(V')}(x)$

which implies that $f^{-1}(R_b^{-1}(V')) = R_a^{-1}(f^{-1}(V'))$. By hypothesis, Ra is a relatively fuzzy continuous mapping from (D, τ_D) to (D, τ_D) and f is a relatively fuzzy continuous mapping from (D, τ_D) to $(f(D), \nu_{f(D)})$.

Therefore $f^{-1}(R_b^{-1}(V')) \cap G = R_a^{-1}(f^{-1}(V')) \cap D$ is open in τ_D. Since f is relatively fuzzy open, then $f(f^{-1}(R_b^{-1}(V')) \cap D) = R_b^{-1}(V') \cap f(D)$ is open in $\nu_{f(D)}$.

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8. References

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